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SUFFICIENT CONDITIONS IN THE CALCULUS OF VARIATIONS.

By W. F. Osgood.

At the time that Weierstrass began his investigations in the Calculus of Variations, the principal necessary conditions had already been obtained. Lagrange had established the differential equation which the solution of the problem must satisfy and Jacobi had determined limitations to the extent of the region over which the integral may be extended. But the question of whether the function thus determined and restricted actually makes the integral a maximum or minimum, and thus yields a solution of the problem, had received no adequate treatment. It was at this point that Weierstrass took up the theory, and he continued its development by supplying sufficient conditions, the establishment of which he based on rigorous analysis. The object of this paper is to give an account of Weierstrass's work. Only the simplest case is considered, and the representation of x and y by means of a parameter t is not introduced. This is desirable, since the gist of Weierstrass's ideas is thus most clearly set forth. Recently Professor Hilbert has simplified the proof of Weierstrass's sufficient conditions, and this proof is also given here.

No previous knowledge of the Calculus of Variations is presupposed. It is regretted that space does not permit a more extensive application of the theory to examples. The reader will find it helpful to apply the principles here treated to the classic problems of this branch of mathematics.*

1. Preliminary Notions and Definitions. The simplest problem of the Calculus of Variations is the following. *To determine a function y of the independent variable x which shall make the integral*

$$I = \int_{x_0}^{x_1} F(x, y, y')dx$$

a maximum or a minimum. Here the function $F(x, y, p)$, together with its partial derivatives of the first and second orders, is a single valued, continuous function of (x, y, p), regarded as independent variables, throughout a region

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* *Cf.* for example Todhunter's *Integral Calculus*, Chap. 15, or, of the older treatises, Jellett, *Calculus of Variations*; Lindelöf-Moigno, *Calcul des variations*; Carll, *Calculus of Variations*. A systematic treatment of the Calculus of Variations from the modern standpoint, illustrated by numerous applications of the theory to classic problems, has just appeared from the pen of Kneser: *Lehrbuch der Variationsrechnung*, Braunschweig, 1900. This is a work of high scientific merit; in point of style, however, it is not all that could be desired, and it is hoped that the present article may make the study of this important treatise easier for those who have not as yet occupied themselves with this field of modern mathematics. Chap. 3 of Kneser's work contains examples that may well be studied in connection with this article, and Chaps. 4 and 5 form an immediate continuation of this article.

$$R: \qquad A_1 < x < B_1, \qquad A_2 < y < B_2, \qquad A_3 < p < B_3,$$

where $A_1 \leqq x_0 < x_1 \leqq B_1$ and some or all of the quantities $A_1, B_1, \ldots B_3$ may be infinite. The limits x_0, x_1 of the integral, for the present, are fixed and the values of the function y_0, y_1 for these values of x are also constant.

Conditions A. The functions $y(x)$ here admitted to consideration shall satisfy the following conditions which will be referred to in what follows as *Conditions A.*

(1) $y(x)$ is a single valued continuous function of x in the interval $x_0 \leqq x \leqq x_1$;

(2) $y(x)$ has a derivative y' at each point of the interval $x_0 < x < x_1$, and $y'(x)$ is a finite and continuous function of x in this interval;

(3) $y(x_0) = y_0$, $y(x_1) = y_1$, where y_0, y_1 are constants for the problem;

(4) the point (x, y, y') lies in the region R.

We shall confine ourselves to the case that the integral I is to be made a minimum. No essential restriction is thereby introduced, since the necessary and sufficient condition that I be a maximum is that $-I$ be a minimum.

Definition of Strong and Weak Variations. Let $y(x)$ be any function of x satisfying Conditions A. Consider a strip S about the curve y, bounded by the curves $y = y(x) + \epsilon$, $y = y(x) - \epsilon$, and by the right lines $x = x_0$, $x = x_1$; ϵ denoting an arbitrarily small positive constant. Let $Y(x)$ be a second function of x also satisfying Conditions A and let the curve $y = Y(x)$ lie wholly in the strip S. Set

$$Y(x) = y(x) + \eta(x).$$

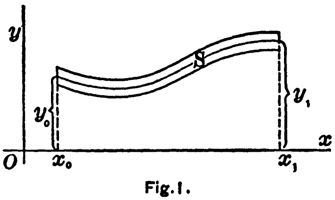

Fig. 1.

The function $\eta(x)$ thus defined is called the *variation of y* and was formerly denoted by δy; the curve $y = Y(x)$ is called the *varied curve.* $\eta(x)$ satisfies nos. (1) and (2) of Conditions A, and $\eta(x_0) = \eta(x_1) = 0$. Moreover,

$$|\eta(x)| < \epsilon, \qquad\qquad x_0 \leqq x \leqq x_1,$$

where $|\eta(x)|$ denotes the absolute (or numerical) value of $\eta(x)$. It does not follow, however, that $\eta'(x)$ is numerically small, too, at all points of the interval. To the special class of variations for which this is the case, — more precisely, for which

$$|\eta'(x)| < \epsilon, \qquad x_0 < x < x_1,$$

the name *weak variations* is applied (Fig. 2); the general variations (which include the above) being termed *strong variations*[*] (Fig. 3).

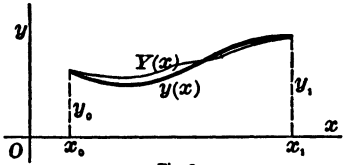

Fig. 2.

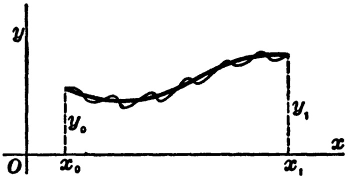

Fig. 3.

By the *neighborhood* of the curve $y = y(x)$ is meant the interior of the strip S together with the extremities of the curve.[†]

Definition of a Minimum. The function $y(x)$ is said to make the integral I a minimum if the value of the integral formed for any varied curve $Y(x) = y(x) + \eta(x)$ has a larger value than when formed for $y(x)$. Writing

$$\Delta I = \int_{x_0}^{x_1} F(x, Y, Y')dx - \int_{x_0}^{x_1} F(x, y, y')dx,$$

we see that the necessary and sufficient condition for a minimum is that $\Delta I > 0$ for all functions $Y(x)$ different from y. The minimum is called a *strong* or a *weak minimum* according as strong or only weak variations are admitted.

Necessary Conditions for a Minimum. Lagrange's Equation. A first necessary condition for a minimum may be obtained as follows. Let $y(x)$ be a function of x satisfying Conditions A and making the integral I a minimum, and let $\eta(x)$ be any variation of y not identically 0. The function $a\eta$, where a is a parameter independent of x and $-1 \leqq a \leqq 1$, is then also a variation. Consider the following function of a:

$$\Delta I = f(a) = \int_{x_0}^{x_1} \{F(x, y + a\eta, y' + a\eta') - F(x, y, y')\}dx.$$

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[*] There is still another class of variations, those denoted by Kneser as *starke Variationen*, which from one point of view is more general than the above; from another point of view this is not the case.

[†] In the case that only weak variations are admitted to consideration — a case that will not concern us in this paper — the definition must be so modified that only such points of S (near the extremities) will be included as can be reached by the varied curves admitted to consideration.

It is a single valued, continuous function of a having a derivative at all points of the interval $-1 \leqq a \leqq 1$, and*

$$f'(a) = \int_{x_0}^{x_1} \{ \eta F_y(x, y+a\eta, y'+a\eta') + \eta' F_{y'}(x, y+a\eta, y'+a\eta') \} \, dx,$$

where $F_y = \frac{\partial F}{\partial y}$, etc. Furthermore, $f(a) > 0$ when $a \neq 0$, since ΔI is then positive, and $f(0) = 0$. Hence $f(a)$ has its minimum value when $a = 0$ and $f'(0) = 0$. We are thus led to the theorem: *A necessary condition for a minimum is that*

$$\int_{x_0}^{x_1} (\eta F_y + \eta' F_{y'}) dx = 0 \tag{1}$$

for all variations η.

The integral standing on the left hand side of (1) is called the *first variation of* I and is denoted by δI.

From this condition it follows that at any point of the curve y at which

$$\frac{\partial^2 F}{\partial y'^2} = F_{y'y'}(x, y, y') \neq 0$$

the function y admits a second derivative and that y satisfies *Lagrange's Equation*:†

$$\left.\begin{aligned} F_y - \frac{dF_{y'}}{dx} &= 0 \\ \text{or} \qquad y'' &= \frac{F_y - F_{xy'} - y'F_{yy'}}{F_{y'y'}} \end{aligned}\right\} \tag{2}$$

Definition of an Extremal. Any function y of x (or curve representing such a function) which, at all interior points of an interval (x', x'') throughout which it is considered satisfies Lagrange's equation (2) is called an *extremal.*‡ The sets of values (x, y, y') thus arising must of course be points of R. In this paper we shall restrict ourselves to the case that $F_{y'y'}(x, y, p)$ does not vanish in R. This case includes nearly or quite all of the problems that have arisen in practice.

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* For a rigorous proof that $f(x)$ has a derivative and that the derivative is given by differentiating under the sign of integration, *cf.* for example Picard, *Traité d'Analyse*, vol. 1, p. 29.

† For a proof of these theorems *cf.* Mr. Whittemore's paper, this number of the ANNALS.

‡ Kneser, *Lehrbuch der Variationsrechnung*, p. 24.

Example.[*] The theory set forth in this paper is well illustrated by the following problem.

To connect two points (x_0, y_0) *and* (x_1, y_1), *of the upper half-plane* (*i. e.* points for which $y_0 > 0$, $y_1 > 0$) *by a curve such that the integral*

$$I = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx,$$

taken along this curve shall be made a minimum.[†]

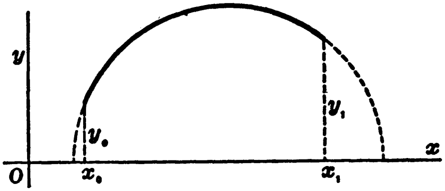

Fig.4.

Here

$$F = \frac{\sqrt{1+y'^2}}{y}, \qquad F_{y'} = \frac{y'}{y\sqrt{1+y'^2}}, \qquad F_{y'y'} = \frac{1}{y(\sqrt{1+y'^2})^3}.$$

Since F does not contain x explicitly, a first integral of Lagrange's equation (2) is given by the formula[‡]

$$F - y'F_{y'} = b^{-1} \text{ (const.)}$$

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[*] This example is continued in §§ 2, 3.

[†] A physical interpretation of this integral may be given as follows. The upper half-plane being conceived as a medium in which the resistance opposed to the motion of a particle is measured by $1/y$, the integral represents the work done against the resistance in moving a particle from the point (x_0, y_0) to the point (x_1, y_1); and the problem is to determine the path along which the work will be a minimum.

This integral presents itself in the non-euclidean geometry in which the Euclidean measurement of angles is preserved, while the lengths of curves are defined by the integral in question; the straight lines being, then, the extremals presently to be obtained. This is one of the non-euclidean geometries that plays a rôle in the uniformizing of the algebraic functions by means of the automorphic functions according to the method of the *Linienelement. Cf.* Klein, *Lineare Differentialgleichungen*, Göttingen, 1894, p. 515, and his lectures on *Automorphic Functions*, summer of 1899; the latter lectures were not lithographed.

[‡] The truth of this theorem is immediately obvious, since in this case

$$\frac{d}{dx}(F - y'F_{y'}) \equiv y'(F_y - \frac{d}{dx}F_{y'}).$$

or
$$\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} = b^{-1},$$
$$y\sqrt{1+y'^2} = b, \quad y'^2 = \frac{b^2-y^2}{y^2},$$
$$\pm\frac{y\,dy}{\sqrt{b^2-y^2}} = dx,$$
$$(x-a)^2 + y^2 = b^2.$$

Hence the extremals are the totality of semicircles that can be drawn in the upper half-plane with their centres on the axis of x. The extremals that go through the point (x_0, y_0) are those for which a and b are connected by the relation
$$(x_0 - a_0)^2 + y_0^2 = b_0^2$$
and they are given by the equation:
$$y = \sqrt{r_0^2 + 2a_0(x-x_0) - x^2}, \tag{3}$$
where $r_0^2 = x_0^2 + y_0^2$, the distance of (x_0, y_0) from the origin. y will go through the second point (x_1, y_1) when*
$$a_0 = \frac{r_1^2 - r_0^2}{2(x_1 - x_0)}$$

Variable Limits of Integration. In the foregoing we have required that the extremities (x_0, y_0), (x_1, y_1) of the curve y for which the integral I is formed be *fixed* points. If these points be allowed to vary, further necessary conditions for a minimum present themselves. Suppose, for example, that the points are required to lie respectively on two continuous curves Γ_0, Γ_1 having tangents that turn continuously. The functions y for which I is to be formed are still subject to Conditions A with the modification that the points (x_0, y_0), (x_1, y_1) are now any points of Γ_0, Γ_1. It may be desirable to demand that the curve y meet the curves Γ_0, Γ_1 only at its extremities. The strip S about a given curve y we may define with Kneser as the part of the x, y plane swept out by a small circular disc of constant radius ρ when its centre describes the curve y, or as the part of such a strip lying between the curves Γ_0, Γ_1 (Fig. 5). The *variation of* y, $\eta(x)$,

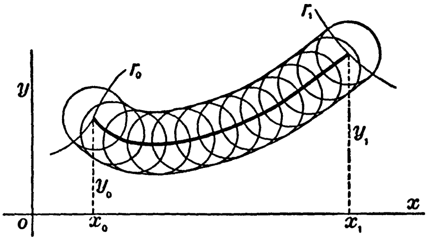

Fig. 5.

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* The case that $x_1 = x_0$ is here excluded. It may be dealt with directly by taking y as the independent variable and its consideration is left to the reader.

is now defined as follows: Consider any *varied curve*, *i. e.* a curve $Y(x)$ satisfying Conditions A and lying wholly in the strip S. Let its extremities be the points (x_0', y_0'), (x_1', y_1'). The interval $x_0' \leqq x' \leqq x_1'$ is next transformed in a one-to-one manner and continuously on the interval $x_0 \leqq x \leqq x_1$, — most simply by the linear transformation

$$x = \frac{x_1' - x'}{x_1' - x_0'} x_0 + \frac{x' - x_0'}{x_1' - x_0'} x_1 .$$

The function $Y(x')$ thus goes over into a function of x for the interval $x_0 \leqq x \leqq x_1$. The variation $\eta(x)$ is defined by the equation

$$Y(x') = y(x) + \eta(x).$$

To obtain the necessary conditions in this case, assume that y with the extremities (x_0, y_0), (x_1, y_1) makes the integral I a minimum and then vary y first by curves Y having the same extremities. It is then shown as in the earlier case that y must satisfy Lagrange's equation.

Assume this condition to be fulfilled. Consider a variable point (x_0', y_0') of Γ_0 and denote the length of the arc of Γ_0 from (x_0, y_0) to (x_0', y_0') by α. Let $y = Y(x', \alpha)$ be the equation of a one-parameter family of varied curves connecting (x_0', y_0') with (x_1, y_1), the latter point being fixed, and let the function $Y(x', \alpha)$ satisfy the further conditions:*

(1) $Y(x', \alpha)$ is, together with its partial derivatives $Y_{x'} = Y'$, Y_α, $Y_{x'\alpha} = Y'_\alpha$, a continuous function of (x', α) in the region $-\alpha_0 \leqq \alpha \leqq \alpha_0$, $x_0' \leqq x' \leqq x_1$

(2) $Y(x', 0) = y(x)$.

Form the function

$$\Delta I = f_1(\alpha) = \int_{x_0'}^{x_1} F(x', Y, Y')dx' - \int_{x_0}^{x_1} F(x, y, y')dx.$$

This function has the same properties as the earlier function $f(\alpha)$, namely: it is positive when $\alpha \neq 0$ and vanishes when $\alpha = 0$, and it has a derivative. Hence it follows that $f_1'(0) = 0$. Compute the derivative.

$$f_1'(\alpha) = \int_{x_0'}^{x_1} \{ F_y Y_\alpha + F_{y'} Y'_\alpha \} dx' - F(x', Y, Y') \Big|_{x' = x_0'} \frac{dx_0'}{d\alpha}$$

This formula can be transformed by means of integration by parts as follows. Notice that $Y_\alpha(x_1, \alpha) = 0$ for all values of α.

$$\int_{x_0'}^{x_1} F_{y'} Y'_\alpha \, dx' = F_{y'} Y_\alpha \Big|_{x_0'}^{x_1} - \int_{x_0'}^{x_1} Y_\alpha \frac{d}{dx'} F_{y'} \, dx',$$

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* It is not difficult to make clear to oneself geometrically that these conditions can all be fulfilled. Kneser (*l. c.* p. 31) gives a simple means of forming an explicit expression for the function $Y(x', \alpha)$, but he does not show that this curve meets Γ_0 in only one point, (x_0', y_0').

$$f_1'(a) = -\left[F_{y'}(x', Y, Y')Y_a + F(x', Y, Y')\frac{dx_0'}{da}\right]_{x'=x_0'}$$
$$+\int_{x_0'}^{x_1}\left\{F_y(x', Y, Y') - \frac{d}{dx'}F_{y'}(x', Y, Y')\right\}Y_a\,dx'.$$

When $a = 0$, $Y = y$ and the integrand in the last formula vanishes identically because of (2). Denote by ω_0 the angle that the tangent to Γ_0 at (x_0, y_0) drawn in the direction of the increasing arc a makes with the axis of x. Then since

$$\frac{dY(x_0', a)}{da} = Y'(x_0', a)\frac{dx_0'}{da} + Y_a(x_0', a) \quad \text{and} \quad \frac{dx_0'}{da}\bigg|_{a=0} = \cos\omega_0, \quad \frac{dY(x_0', a)}{da}\bigg|_{a=0} = \sin\omega_0,$$
$$Y_a(x_0, 0) = \sin\omega_0 - y_0'\cos\omega_0$$

The condition $f_1'(0) = 0$ thus leads to the following result. *A further necessary condition in the case of variable limits of integration is that*

$$[F(x_0, y_0, y_0') - y_0'F_{y'}(x_0, y_0, y_0')]\cos\omega_0 + F_{y'}(x_0, y_0, y_0')\sin\omega_0 = 0. \tag{4}$$

A similar formula holds for the point (x_1, y_1) and is obtained by replacing the subscript 0 by 1. These two formulas usually suffice to determine an extremal completely.

When an extremal y is cut by a curve Γ_0 at such an angle that equation (4) is satisfied, y is said to be cut *transversely* by Γ_0 (Kneser).

2. The Notion of a Field about the Extremal C.

If there is to exist a function y which makes the integral I a minimum, then it must be possible, first of all, to join the points A and B (*i. e.* (x_0, y_0) and (x_1, y_1)) by an extremal C. We will assume that this condition is fulfilled. This will always be the case when Lagrange's equation (2), which is a differential equation of the second order, admits in the interval $x_0 \leqq x \leqq x_1$ a solution $g(x, a, b)$ depending in such a manner on two arbitrary constants a, b that the equations

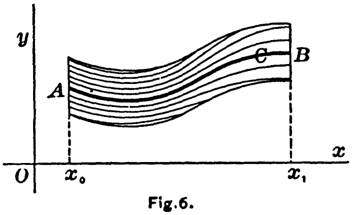

Fig. 6.

$$y_0 = g(x_0, a, b), \qquad y_1 = g(x_1, a, b)$$

can be solved for a, b.

Secondly, we assume that a one-parameter family of extremals,

$$y = \phi(x, \gamma),$$

exists, one of which, $y = \phi(x, \gamma_0)$, coincides with C, and which sweep out the neighborhood of C just once; and that furthermore the distance between two neighboring extremals of this family, measured along a parallel to the y-axis, is an infinitesimal of the same order throughout the whole interval (x_0, x_1). More precisely, we assume* :

(1) *that a function* $\phi(x, \gamma)$ *exists which, together with its partial derivatives* $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi'_\gamma$, *is a continuous function of the two independent variables* (x, γ) *in the domain*

$$T: \qquad x_0 \leqq x \leqq x_1, \qquad \gamma_0 - \kappa \leqq \gamma \leqq \gamma_0 + \kappa,$$

where κ *is a suitably chosen positive constant; and that, for a constant value of* γ, $\phi(x, \gamma)$ *is an extremal, which for the special value* $\gamma = \gamma_0$, *coincides with* C;

(2) *that* $\phi_\gamma(x, \gamma_0) \neq 0$ *when* $x_0 \leqq x \leqq x_1$.

From (1) and (2) it follows, as will presently be shown, that through each point of a properly chosen neighborhood S of C one and only one extremal $\phi(x, \gamma)$ passes and that $\phi_\gamma(x, \gamma)$ does not vanish at any point of S. Hence the extremals $\phi(x, \gamma)$ pertaining to T just fill, when κ is chosen sufficiently small, a two-dimensional region S' in which the extremal C is imbedded. The picture of the cross-section of the strata of a geological formation is useful here, while in the case of two independent variables† (I being a double integral and the extremals $\phi(x, y, \gamma)$ a family of surfaces) the notion of the strata themselves serves to illustrate well the relations in question.

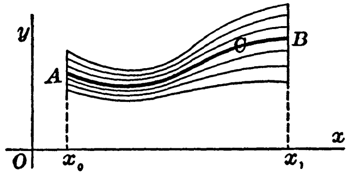

Fig.7.

Proof. We will first give the proof, not for the neighborhood S of C, but for the neighborhood of an arbitrary point (x', y') of C. Since $\phi_\gamma(x, \gamma)$ is continuous and does not vanish at the point (x', γ_0), there exists a certain neighborhood $|x - x'| < h'$, $|\gamma - \gamma_0| < \kappa'$ about this point, throughout which $\phi_\gamma(x, \gamma) \neq 0$. Furthermore the equation

$$y = \phi_\gamma(x, \gamma)$$

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* A broader definition of the field than this may be given; *cf.* the writer's paper: "On the Existence of a Minimum, etc.," *Transactions Amer. Math. Soc.*, vol. 2 (1901).

† *Cf.* Schwarz, *Ueber ein die Flächen kleinsten Inhalts betreffendes Problem der Variationsrechnung*, Festschrift on the occasion of Weierstrass's 70th birthday, 1885, *Werke*, vol. 1, p. 224, where the notion of the field, which is the foundation on which Weierstrass builds up his sufficient condition for a minimum, is set forth with admirable clearness.

can be solved for γ as a function of the two independent variables (x, y), the domain of these variables being the neighborhood $|x - x'| < h'$, $|y - y'| < h'$ of the point (x', y'). The proof follows at once from the fundamental existence theorem regarding implicit functions,* which for the present case may be stated as follows.

Let $\Phi(x, y, \gamma)$ *be a continuous function of the three independent variables* (x, y, γ) *throughout the neighborhood* D *of the point* (x', y', γ_0) *and let*

$$\Phi(x', y', \gamma_0) = 0;$$

let the first partial derivatives of Φ *exist at each point of* D *and be continuous, and let*

$$\Phi_\gamma(x', y', \gamma_0) \neq 0;$$

then for every point (x, y) *of the neighborhood* d *of* (x', y') *the equation*

$$\Phi(x, y, \gamma) = 0$$

has one and only one root γ *that lies in the neighborhood of the point* $\gamma = \gamma_0$, *and thus defines* γ *as a single valued function of* (x, y):

$$\gamma = \psi(x, y).$$

The function $\psi(x, y)$ *is continuous throughout* d *and has continuous first partial derivatives there, given by the usual formulas for the differentiation of implicit functions:*

$$\Phi_x + \Phi_\gamma \cdot \frac{\partial \gamma}{\partial x} = 0, \qquad \Phi_y + \Phi_\gamma \frac{\partial \gamma}{\partial y} = 0.$$

In the present case we have merely to set

$$\Phi(x, y, \gamma) = \phi(x, \gamma) - y.$$

Then $\Phi_\gamma(x, y, \gamma) = \phi_\gamma(x, \gamma)$. Thus the equation $y = \phi(x, \gamma)$ has a singly determinate solution, $\gamma = \psi(x, y)$, and the proof is complete for the neighborhood $|x - x'| < h'$, $|y - y'| < h'$, $|\gamma - \gamma_0| < \kappa'$ of the point (x', y', γ_0).

The quantities h', κ' will in general vary for different points of the curve

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* This theorem and its proof were first given by Dini in his university lectures at Pisa, *Analisi infinitesimale*, 2 vols., 1877–78, lithographed, vol. 1. They were printed in Peano-Genocchi's *Calcolo differenziale e principii di calcolo integrale*, Turin, 1884, §110, and appeared later in Jordan's *Cours d'Analyse*, 2nd ed., vol. 1, 1893, §91.

C; but if they are chosen suitably at each point (x', y'), the lower limit of each of them, as (x', y') describes C, will be a positive quantity, as can be shown at once by a well known method of reasoning in higher analysis. This completes the proof.

Example. Consider the example of §1. The extremal C is a semicircle of radius b_0 with its centre on the axis of x at the point $x = a_0$. A field can be formed about C in an unlimited number of ways. Thus we may take as the family of extremals the family of concentric semicircles and set

$$\phi(x, \gamma) = \sqrt{\gamma^2 - (x - a_0)^2}, \qquad 0 < \gamma < \infty, \tag{5}$$

Here, $\gamma_0 = b_0$. Or, again, let (x'_0, y'_0) be an arbitrary point of the extremal C extended into the region $x < x_0$, so that $a_0 - b_0 < x'_0 < x_0$, (Fig. 8). A field will be formed by the semicircles through (x'_0, y'_0) with their centres on the axis of x, and

$$\phi(x, \gamma) = \sqrt{r_0'^2 + 2\gamma(x - x'_0) - x^2}, \qquad -\infty < \gamma < +\infty, \tag{6}$$

where $r_0'^2 = x_0'^2 + y_0'^2$. Here, $\gamma_0 = a_0$.

3. Weierstrass's Sufficient Condition for a Minimum; the Function $\mathcal{E}$.

In order to establish the existence of a minimum, Weierstrass gives a direct proof that, when a field about C exists in the sense of §2 and certain further conditions are fulfilled, the integral I, taken along the extremal C, has a smaller value, J, than the value $\bar{J}$ which it acquires when taken along any other curve $\bar{C}$ corresponding to a function $Y(x)$ (§1) not identical with $y(x)$.

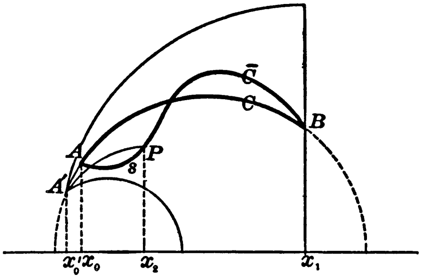

Fig. 8.

We will assume, then, that such a field exists and, for simplicity, that all the extremals $\phi(x, \gamma)$ go through a point A', (x'_0, y'_0), on the extremal C produced, where $x'_0 < x_0$, but lies arbitrarily near to x_0.* Let P, (x_2, y_2), be any point of the curve $\bar{C}$ and s the length of the arc AP of $\bar{C}$. Denote the total

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*Weierstrass considered in his lectures the case of a field in which the extremals $\phi(x, \gamma)$ all pass through the point (x_0, y_0). The notion of the auxiliary point (x'_0, y'_0) is due to Kneser (*l. c.*, p. 59). Its introduction in the presentation of Weierstrass's proof simplifies the details of that proof without altering its substance.

length of $\bar{C}$ by l. Draw the extremal $y = \phi(x, \gamma_2)$ connecting A' and P. Consider the integrals:

$$I_{02} = \int_{x_0'}^{x_2} F(x, y, y')\,dx \qquad \text{taken along } y = \phi(x, \gamma_2),$$

$$\bar{I}_{21} = \int_{x_2}^{x_1} F(x, Y, Y')\,dx \qquad \text{taken along } \bar{C}.$$

The sum of these integrals is a function of s. Denote it by $f(s)$:

$$f(s) = I_{02} + \bar{I}_{21}.$$

Observe that $f(l) = I' + J$, the value of I taken along C from (x_0', y_0') to (x_1, y_1); $f(0) = I' + \bar{J}$, the value of I taken along C from (x_0', y_0') to (x_0, y_0) and then along $\bar{C}$ from (x_0, y_0) to (x_1, y_1); so that

$$f(l) - f(0) = -(\bar{J} - J).$$

LEMMA. *The function $f(s)$ is a continuous function of s in the interval $0 \leqq s \leqq l$ and it has a derivative that is never positive and is sometimes negative, provided that the function*

$$F_{y'y'}(x, y, p)$$

is positive at every point (x, y) of C when to the third argument p an arbitrary value is assigned.

From this theorem it follows that $f(s)$ is a monotonic function that actually decreases in parts of the interval $0 \leqq s \leqq l$, and hence that

$$f(l) < f(0), \qquad i.\,e. \quad J < \bar{J}.$$

The foregoing conditions are thus seen to be sufficient for the existence of a minimum.

*Proof of the Lemma.** First, it is clear that

$$\frac{d\bar{I}_{21}}{ds} = -F(x_2, y_2, M)\frac{dx_2}{ds} = -\cos\alpha\, F(x_2, y_2, M),$$

where M denotes the slope of $\bar{C}$ at P and $\alpha = \tan^{-1}M$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

---

* Weierstrass obtains the derivative $f'(s)$ by computing directly the infinitesimal increment of the function in terms of the infinitesimal increment of s and evaluating the limit of the ratio. The analytic work is much longer, even when Weierstrass's parametric representation of x and y in terms of t is replaced by the unsymmetric form here employed, and appears to possess no compensating advantages.

The integral I_{02} may be regarded as a function of the independent variables (x_2, γ) and these in turn each as a function of s. Then

$$\frac{dI_{02}}{ds} = \frac{\partial I_{02}}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial I_{02}}{\partial \gamma}\frac{d\gamma}{ds}.$$

$$\frac{\partial I_{02}}{\partial x_2}\frac{dx_2}{ds} = \cos a\, F(x_2, y_2, m),$$

where m denotes the slope of C at P.

The derivative $\frac{\partial I_{02}}{\partial \gamma}$ can be obtained by differentiating under the sign of integration:*

$$\frac{\partial I_{02}}{\partial \gamma} = \int_{x_0}^{x_2} (F_y\, \phi_\gamma + F_{y'}\, \phi'_\gamma)\, dx,$$

and this integral can be transformed by an application of the method of integration by parts similar to that employed in §1. The result is as follows:

$$\frac{\partial I_{02}}{\partial \gamma} = F_{y'}(x_2, y_2, m)\, \phi_\gamma(x_2, y_2) + \int_{x_0}^{x_2} (F_y - \frac{d}{dx} F_y)\, \phi_\gamma dx$$

The integral on the right hand side vanishes because y satisfies Lagrange's equation (2).

To compute the derivative $d\gamma/ds$, observe that, the equation

$$y = \phi(x, \gamma)$$

being solved for γ (*cf.* §2):

$$\gamma = \psi(x, y),$$

the partial derivatives ψ_x, ψ_y are given by the formulas

$$\psi_x = -\frac{\phi'}{\phi_\gamma}, \qquad \psi_y = \frac{1}{\phi_\gamma}.$$

If now (X, Y) denote the coordinates of a variable point of $\bar{C}$,

$$\gamma_2 = \psi(X, Y)$$

and

$$\frac{d\gamma_2}{ds} = \psi_x(X, Y) \cos a + \psi_y(X, Y) \sin a$$

$$= \frac{-\phi'(x_2, y_2) \cos a + \sin a}{\phi_\gamma(x_2, y_2)} = \cos a \frac{M - m}{\phi_\gamma(x_2, y_2)}$$

---

* Picard, *l. c.* The differentiation of I_{02} may also be performed by the formula established by Picard on p. 31.

Collecting the results we have the following formulas:

$$\frac{dI_{02}}{ds} = \cos a\ [F(x_2, y_2, m) + (M-m)F_{y'}(x_2, y_2, m)],$$

$$f'(s) = -\mathcal{E},$$

where

$$\left.\begin{aligned} \mathcal{E} &= \cos a\ \{F(x_2, y_2, M) - F(x_2, y_2, m) - (M-m)F_{y'}(x_2, y_2, m)\} \\ &= \tfrac{1}{2}\cos a\ (M-m)^2 F_{y'y'}(x_2, y_2, m + \theta(M-m)), \quad 0 < \theta < 1. \end{aligned}\right\} \quad (7)$$

Thus the existence of a derivative and with it the continuity of the function $f(s)$ has been established at all points of the interval $0 \leqq s \leqq l$.

From the form of $\mathcal{E}$ it appears that $f'(s)$ can never be positive when $F_{y'y'}(x, y, p)$ satisfies the conditions of the lemma; but it can vanish when $M - m = 0$, *i. e.* when the extremal $\phi(x, \gamma_2)$ is tangent to $\bar{C}$ at P. Since $\bar{C}$ is arbitrary, it may coincide throughout parts of the interval with an extremal $\phi(x, \gamma)$, and here $M - m$ will vanish. It remains to show that $M - m$ cannot vanish throughout the entire interval unless $\bar{C}$ coincides throughout with C. This follows at once from the equation

$$\frac{d\gamma_2}{ds} = \cos a\ \frac{M - m}{\phi_\gamma(x_2, \gamma_2)};$$

for if $M - m \equiv 0$, then γ_2 is constant, and since only one extremal C can be drawn in S connecting A and B, $\gamma_2 = \gamma_0$.

The method here set forth for a special field admits of extension to the general field of §2 (*cf.* §§4, 5) and the results may be summarized as follows:

WEIERSTRASS'S SUFFICIENT CONDITION FOR A MINIMUM. *A sufficient condition that a function y make the integral*

$$I = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

a minimum is

(1) *that the points (x_0, y_0), (x_1, y_1) can be joined by an extremal C satisfying Conditions A;*

(2) *that there exists a field about C in the sense of §2;*

(3) *that $F_{y'y'}(x, y, p)$ be positive at all points of C, p being arbitrary*

in the case of a strong minimum, and equal to y' in the case of a weak minimum.

Example. The integral of the example of §1 is a minimum when the function y is an extremal. For, if the field designated at the close of §2 is employed, Weierstrass's sufficient condition will be fulfilled.

4. Proof of the Sufficient Condition for Any Field; Kneser's Transversal. Let it be assumed that $A_1 < x_0$ (§1, beginning) and that the field extends beyond the line $x = x_0$, the region T being now enlarged slightly to a region

$$T': \qquad x_0' \leqq x \leqq x_1, \qquad \gamma_0 - \kappa \leqq \gamma \leqq \gamma_0 + \kappa,$$

where $A_1 < x_0' < x_0$, but where x_0' may be chosen arbitrarily near to x_0.

Through the point A pass an arbitrary curve G not tangent to C and having a continuously turning tangent. Let the coordinates of a variable point Q of G be (ξ, η). Then, as is readily shown, ξ and η are both continuous functions of γ having continuous derivatives with regard to γ, and these derivatives do not vanish simultaneously in the neighborhood of the point A. For

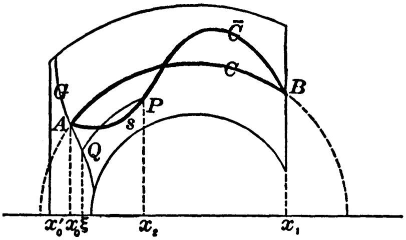

Fig. 9.

$$\frac{d\eta}{d\gamma} = \phi'(\xi, \gamma)\, \frac{d\xi}{d\gamma} + \phi_\gamma(\xi, \gamma), \tag{8}$$

and $\phi_\gamma(x_0, \gamma_0) \neq 0$.

Let the points A and B be joined as before by an arbitrary curve $\bar{C}$ satisfying Conditions A and lying in the neighborhood of C, and let the definitions and notation of §3 be preserved with the one exception that I_{02} shall now denote the following integral:

$$I_{02} = \int_{\xi}^{x_2} F(x, y, y')\, dx \text{ taken along the extremal } y = \phi(x, \gamma).$$

Then
$$f(s) = I_{02} + \bar{I}_{21},$$
$$f'(s) = \frac{dI_{02}}{ds} + \frac{d\bar{I}_{21}}{ds},$$
$$\frac{d\bar{I}_{21}}{ds} = -F(x_2, y_2, M) \cos a,$$

$$\frac{dI_{02}}{ds} = \frac{\partial I_{02}}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial I_{02}}{\partial \xi}\frac{d\xi}{ds} + \frac{d\gamma_2}{ds}\int_{\xi}^{x_2}\left\{F_y\,\phi_\gamma + F_{y'}\,\phi'_\gamma\right\}dx$$
$$= F(x_2, y_2, m) \cos a - \frac{d\gamma_2}{ds}\left[F(\xi, \eta, \eta_1)\frac{d\xi}{d\gamma} + F_{y'}(\xi, \eta, \eta_1)\,\phi_\gamma(\xi, \gamma)\right]_{\gamma=\gamma_2}$$
$$+ \frac{d\gamma_2}{ds} F(x_2, y_2, m)\,\phi_\gamma(x_2, \gamma_2)$$

where $\eta_1 = \phi'(\xi, \gamma)$, the residual integral vanishing as in §3. We thus obtain the formula

$$f'(s) = -\cos a\,[F(x_2, y_2, M) - F(x_2, y_2, m) - (M - m)\,F_{y'}(x_2, y_2, m)]$$
$$- \frac{d\gamma_2}{ds}\left[F(\xi, \eta, \eta_1)\frac{d\xi}{d\gamma} + F_{y'}(\xi, \eta, \eta_1)\,\phi_\gamma(\xi, \gamma)\right]_{\gamma=\gamma_2}.$$

This formula differs from the corresponding one of §3, $f'(s) = -\mathcal{E}$, only by the presence of the second line. The curve G is still arbitrary; and now it is Kneser's[*] idea so to determine G that this line will drop out:

$$F(\xi, \eta, \eta_1)\frac{d\xi}{d\gamma} + F_{y'}(\xi, \eta, \eta_1)\,\phi_\gamma(\xi, \gamma) = 0 \tag{9}$$

This will always be possible provided $F(x_0, y_0, y'_0) \neq 0$.[†] Thus ξ is defined

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[*] *l. c.* ch. 3. The further course of Kneser's proof follows, however, different lines from those of this article, which are in substance Weierstrass's. Kneser treats the whole problem of making the integral I a minimum as a generalization of the problem of the shortest lines on a curved surface.

[†] If $F(x_0, y_0, y'_0) = 0$ we may consider the function $\bar{F}(x, y, y') = F(x, y, y') + 1$. Then

$$\int_{x_0}^{x_1}\bar{F}(x, y, y')dx = \int_{x_0}^{x_1}F(x, y, y')dx + (x_1 - x_0).$$

The functions F, $\bar{F}$ have the same extremals and evidently a function y that makes one a minimum makes the other a minimum too. But $\bar{F}(x_0, y_0, y'_0) \neq 0$, and the proof of the text applies to it. Hence even when $F(x_0, y_0, y'_0) = 0$, y makes I a minimum.

by (9) as a function of γ, single valued and continuous and having a continuous derivative in the neighborhood of the point $\gamma = \gamma_0$,* and $\eta = \phi(\xi, \gamma)$. The curve is transverse (§1) to the extremals $\phi(x, \gamma)$; for from (8) and (9) follows that

$$[F(\xi, \eta, \eta_1) - \eta_1 F_{y'}(\xi, \eta, \eta_1)] \frac{d\xi}{d\gamma} + F_{y'}(\xi, \eta, \eta_1) \frac{d\eta}{d\gamma} = 0$$

With the modification, then, that the integral I_{02} is to be taken from the lower limit ξ just determined as a function of γ, instead of from the point (x_0', y_0') of §3, the reasoning of that section applies word for word to the present case, and thus Weierstrass's sufficient condition is established for any field.

Hitherto we have confined ourselves to the case of fixed limits of integration. Kneser's method of proof enables us at once to extend the sufficient condition to the case of variable limits and enunciate the

THEOREM. *A sufficient condition in the case of variable limits of integration that the extremal C make the integral I a minimum is that* (the necessary conditions that C cut Γ_0, Γ_1 transversely being fulfilled) *it be possible to surround C by a field of extremals in the sense of* §2 *and that each of the extremals of this field cut the curve Γ_0* (§1) *transversely; and finally that the function $F_{y'y'}$ be positive in R.*

In §2 the field was bounded by the right lines $x = x_0$ and $x = x_1$. Here, those lines are replaced by the curves Γ_0, Γ_1.

5. Hilbert's Proof of the Sufficient Conditions by Means of an Invariant Integral. It not infrequently happens that a problem in physics can be solved independently by the methods of the differential calculus and by those of the integral calculus. Thus the pressure P of a fluid, acted on only by the force of gravity, on an immersed vertical plane area can be obtained by determining dP/dx as a function of x, the depth, and then solving the corresponding differential equation; or P can be expressed as the limit of the sum of the partial pressures ΔP on horizontal strips of the area of infinitesimal breadth. Or, again, the condition for a steady flow of heat in a homogeneous solid—Laplace's equation—can be obtained by considering the amount of heat that flows into an infinitesimal element of the body and making the linear dimensions of this element converge toward zero; or Green's theorem can be applied to an arbitrary portion of the body, which is not ultimately made infinitesimal.†

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* For the existence theorem for an ordinary differential equation, of which use is here made, *cf.* for example Picard, *Traité d'Analyse*, vol. 2, ch. 11, in particular, §7.

† A recent illustration of these two methods was afforded by the two proofs of Goursat's lemma for Cauchy's Integral Theorem, the proof that Goursat gave (*cf. Transactions of the Amer. Math. Soc*, vol. 1 (1900), p. 14), corresponding to the integral method, while Moore's proof (*ibid.*, p. 499) corresponds to the differential method.

Weierstrass's method of establishing his sufficient condition was a method of the differential calculus. Hilbert's method,* which we now proceed to consider, is a method of the integral calculus. The problem is (*cf.* §3) to show that the value J of the integral I taken along the extremal C is less than its value $\bar{J}$ taken along an arbitrary curve $\bar{C}$ lying in the neighborhood of C. Hilbert proposes to himself this question: *Is it possible to determine an integral:*

$$\int_{x_0}^{x_1} \Phi(x, Y, Y')dx \tag{10}$$

in such a manner

(*a*) *that its value shall be invariant of the curve $\bar{C}$, i. e. of the function $Y(x)$ which is merely required to satisfy Conditions A and to lie in the neighborhood of C;*

(*b*) *that, when $\bar{C}$ coincides with C, i. e. $Y(x) = \phi(x, \gamma_0)$, the value of this integral shall reduce to the value of I taken along C, i. e. to J.*

Suppose such an integral has once been obtained. Its value is then J. And now the point of Hilbert's whole method is this. The difference, $\bar{J} - J$, which for a minimum must be a positive quantity (necessary and sufficient condition, §3) can be written in the form

$$\bar{J} - J = \int_{x_0}^{x_1} \{F(x, Y, Y') - \Phi(x, Y, Y')\} dx, \tag{11}$$

and *hence a sufficient condition for a minimum is that the integrand*

$$F(x, Y, Y') - \Phi(x, Y, Y')$$

be positive in parts of the interval, and nowhere negative, for all curves $\bar{C}$ not coincident throughout with C. It turns out that, in case a field about the extremal C exists, the function Φ can be so chosen that this integrand becomes, save as to a positive factor, the Weierstrassian function $\mathcal{E}$.

We proceed now to develop the details of Hilbert's analysis, and begin by citing, for the purposes of a lemma, a well known theorem of the calculus of variations.

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* Hilbert set forth his method in a course of lectures held at Göttingen in the summer semester of 1900; *cf.* the report on these lectures published in the *Bericht des Mathematischen Vereins*, Göttingen, *S.-S.*, 1900, p. 10.

LEMMA.* *The necessary and sufficient condition that the value of the integral*

$$\int_{x_0}^{x_1} \Phi(x, y, y')\,dx,$$

where Φ is subject to the same conditions as the function F, §1, be independent of the path y, which is required merely to satisfy Conditions A and, besides, have a finite and continuous second derivative in the interval $x_0 < x < x_1$,† is that Φ satisfy the functional equation

$$\Phi_y - \frac{d}{dx}\Phi_{y'} = 0 \tag{12}$$

for all functions y of the above class.

This equation is, in form, identical with Lagrange's equation (2). But there, F was given and y was required to satisfy the equation; here, y is assumed at pleasure and Φ is required to satisfy the equation.

The special case of the theorem that $\Phi = P + y'Q$, where P, Q are functions of only (x, y), the integral then reducing to

$$\int_{(x_0, y_0)}^{(x_1, y_1)} P\,dx + Q\,dy,$$

is a familiar one. Equation (12) then becomes

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The application that Hilbert makes of the lemma is as follows. He takes as Φ the function

$$\Phi(x, y, y') = F(x, y, p) + (y' - p)F_p(x, y, p), \tag{13}$$

where p denotes an arbitrary function of (x, y) having continuous first partial derivatives, p_x, p_y, at all points of the field, and then requires that p shall be so restricted that Φ will satisfy (12). Here,

$$\Phi_y = F_y + F_p p_y + (y' - p)(F_{yp} + F_{pp} p_y) - F_p p_y,$$
$$\frac{d\Phi_{y'}}{dx} = \frac{dF_p}{dx} = F_{xp} + F_{yp}\,y' + F_{pp}(p_x + p_y y'),$$

---

* A proof of this lemma is given at the close of the present section.

† This further requirement of the existence of a second derivative must in general be made for the sufficient condition in order that (12) may have a meaning. But when Φ is a linear function of y', as in the case considered by Hilbert, the requirement is always fulfilled.

and the condition for p is found to be :*

$$F_y - F_{xp} - p\,F_{yp} - F_{pp}(p_x + p\,p_y) = 0. \tag{14}$$

This last condition can always be satisfied when a field exists by choosing for p the function :

$$p = \phi'(x, \gamma), \qquad y = \phi(x, \gamma). \tag{15}$$

In fact, (14) then reduces to Lagrange's equation (2) ; for

$$p_x = \phi'' + \phi'_\gamma \frac{\partial \gamma}{\partial x}, \qquad p_y = \phi'_\gamma \frac{\partial \gamma}{\partial y},$$

$$0 = \phi' + \phi_\gamma \frac{\partial \gamma}{\partial x}, \qquad 1 = \phi_\gamma \frac{\partial \gamma}{\partial y} ;$$

and so

$$p_x + p\,p_y = \phi''.$$

Replacing now, in the expression for Φ, (13), y by Y and p by $y' = \phi'(x, \gamma)$, where $Y = \phi(x, \gamma)$, equation (11) becomes the following (*cf.* (7)) :

$$\bar{J} - J = \int_{x_0}^{x_1} \{ F(x, Y, Y') - F(x, Y, y') - (Y' - y')\,F_{y'}(x, Y, y') \} dx$$

$$= \int_{x_0}^{x_1} \mathcal{E} \sec \alpha \, dx = \int_{s_0}^{s_1} \mathcal{E} \, ds, \tag{16}$$

where ds denotes an element of the curve $\bar{C}$.

Proof of the Lemma. (*a*) The condition is necessary. For let Y, y be any two of the functions, and let $Y = y + \eta$. Form the function

$$f(\alpha) = \int_{x_0}^{x_1} \{ \Phi(x, y + \alpha\eta, y' + \alpha\eta') - \Phi(x, y, y') \} dx.$$

Since $y + \alpha\eta$, α being a constant, is also one of the functions considered, $f(\alpha) = 0$ and hence

$$f'(\alpha) = \int_{x_0}^{x_1} \{ \eta\,\Phi_y(x, y + \alpha\eta, y' + \alpha\eta') + \eta'\,\Phi_{y'}(x, y + \alpha\eta, y' + \alpha\eta') \} dx$$

also vanishes. From the equation obtained by putting $\alpha = 0$:

---

* This condition is obtained by Hilbert in the form (*cf. Bericht des Math. Vereins, l. c.*)

$$\frac{\partial F_p}{\partial x} - \frac{\partial F}{\partial y} + \frac{\partial}{\partial y}(p\,F_p) = 0,$$

where the partial differentiations indicated by the sign ∂ are performed on the supposition that (x, y) are the independent variables.

$$\int_{x_0}^{x_1} \left\{ \eta\, \Phi_y(x, y, y') + \eta'\, \Phi_{y'}(x, y, y') \right\} dx = 0$$

equation (12) can be deduced by the method set forth in Mr. Whittemore's paper (*l. c.*). Hence the condition is necessary.

(*b*) Conversely, the condition is sufficient. Again, let Y, y be any two of the functions and form the function $f(\alpha)$ and its derivative $f'(\alpha)$. The integral that expresses this latter function can be transformed by integration by parts, as in §1, so that

$$f'(\alpha) = \int_{x_0}^{x_1} \eta \left\{ \Phi_y(x, y + \alpha\eta, y' + \alpha\eta') - \frac{d}{dx}\, \Phi_{y'}(x, y + \alpha\eta, y' + \alpha\eta') \right\} dx.$$

Hence $f'(\alpha) \equiv 0$ and (12) is a sufficient condition.

6. Extension of Hilbert's Method to Double Integrals; the Function $\mathcal{E}$. The problem with which this paper opens can be extended immediately to double integrals. Let $\mathfrak{T}$ be a region of the (x, y)-plane bounded by a closed curve $\mathfrak{C}$ which has a continuously turning tangent, and let an arbitrary set of boundary values $\mathfrak{b}$ be given along $\mathfrak{C}$, these values to form a continuous function of the arc of $\mathfrak{C}$ and to admit a continuous derivative with regard to this arc. Consider the integral

$$I = \iint\limits_{\mathfrak{T}} F(x, y, z, p, q)\, dx dy \qquad \left(p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right) \tag{17}$$

extended over $\mathfrak{T}$, z being a continuous function of (x, y) throughout $\mathfrak{T}$ inclusive of the boundary and assuming on the boundary the values $\mathfrak{b}$. Concerning the continuity of F and its derivatives of the first and second orders throughout a region $\mathfrak{R}$, a similar assumption to that of §1 is made, while z is required to have partial derivatives of the first and second* orders continuous and finite within $\mathfrak{T}$.

The problem is to show that there exists among these functions z one which makes I a minimum.

The problem may be stated geometrically as follows: On the elements of the cylinder whose generatrix is the curve $\mathfrak{C}$ and whose elements are parallel to the z-axis, lay off the boundary values. Thus a tortuous curve $\mathfrak{c}$ is generated. And now it is required to span a surface into $\mathfrak{c}$ which shall make the integral I a minimum.

First, necessary conditions are established. Assume the problem solved and let z be the solution. Let $Z = z + \zeta$ be any *varied surface*, ζ the *variation* of z. Then it is shown that the relation

---

* The assumption of the existence of second partial derivatives has not, so far as I know, as yet been shown to be superfluous in the case of the double integral.

$$\iint\limits_{\mathfrak{T}} \{ \zeta F_z + \pi F_p + \kappa F_q \} \, dx dy = 0 \qquad \left(\pi = \frac{\partial \zeta}{\partial x}, \ \kappa = \frac{\partial \zeta}{\partial y} \right)$$

must hold, no matter what variation ζ be chosen. The assumption of the existence of second partial derivatives of z is employed in the next step, which consists in transforming this integral by the method of integration by parts (as in §1, end), in order to obtain *Lagrange's Equation*:

$$\left. \begin{array}{rl} & F_z - \dfrac{\partial F_p}{\partial x} - \dfrac{\partial F_q}{\partial y} = 0 \\ \text{or} & F_z - F_{xp} - F_{yq} - F_{zp} \dfrac{\partial z}{\partial x} - F_{zq} \dfrac{\partial z}{\partial y} \\ & - \left(F_{pp} \dfrac{\partial^2 z}{\partial x^2} + 2 F_{pq} \dfrac{\partial^2 z}{\partial x \partial y} + F_{qq} \dfrac{\partial^2 z}{\partial y^2} \right) = 0 \end{array} \right\} \qquad (18)$$

It is further shown that, in the case of a minimum, the expression

$$F_{pq}^2 - F_{pp} F_{qq} \leqq 0$$

at every point of the surface z, (*Legendre's Condition*). We will assume that

$$F_{pq}^2 - F_{pp} F_{qq} < 0 \qquad (19)$$

at all points of $\mathfrak{R}$.

Suppose, now, that we have succeeded in showing that a function z exists, fulfilling the conditions of continuity specified above, taking on the prescribed boundary values, and satisfying Lagrange's equation throughout $\mathfrak{T}$. This function is the extremal C. What more is requisite in order that this function may make I a minimum? It turns out that Weierstrass's sufficient condition of §3 can be extended to this case immediately. First, then, we shall require the existence of a *field* about the extremal C. We assume:

(1) *that a function $\phi(x, y, \gamma)$ exists which, together with its partial derivatives $\phi_x, \phi_y, \phi_\gamma, \phi_{x\gamma}, \phi_{y\gamma}$, is a continuous function of the three independent variables (x, y, γ) in the domain*
T: (x, y), a point of $\mathfrak{T}$, inclusive of $\mathfrak{C}$; $\gamma_0 - \kappa \leqq \gamma \leqq \gamma_0 + \kappa$, where κ is a positive constant; and that, for a constant value of γ, $\phi(x, y, \gamma)$ is an extremal, which for the special value $\gamma = \gamma_0$ coincides with C;

(2) *that $\phi_\gamma(x, y, \gamma_0) \neq 0$ when (x, y) lies in $\mathfrak{T}$ or on $\mathfrak{C}$.*

In order to obtain the Weierstrassian function $\mathcal{E}$, we apply Hilbert's method. First comes the

LEMMA. *The necessary and sufficient condition that the value of the integral*

$$\iint\limits_{\mathfrak{T}} \Phi(x, y, z, p, q)\,dx\,dy$$

be independent of the surface z is that

$$\Phi_z - \frac{\partial \Phi_p}{\partial x} - \frac{\partial \Phi_q}{\partial y} = 0 \tag{20}$$

Here, all surfaces considered are spanned into the same tortuous curve $\mathfrak{c}$, and Φ and z are subjected to the same conditions of continuity as F and z above. The differentiations indicated by the round ∂ are performed on the supposition that (x, y) are the independent variables.

Next, a function Φ is chosen corresponding to (13):

$$\begin{aligned}\Phi(x, y, z, p, q) = F(x, y, z, \mathfrak{p}, \mathfrak{q}) &+ (p - \mathfrak{p})\, F_p(x, y, z, \mathfrak{p}, \mathfrak{q}) \\ &+ (q - \mathfrak{q})\, F_q(x, y, z, \mathfrak{p}, \mathfrak{q})\end{aligned} \tag{21}$$

where $\mathfrak{p}$, $\mathfrak{q}$ denote arbitrary functions of (x, y, z) having continuous first partial derivatives; and then it is required that $\mathfrak{p}$, $\mathfrak{q}$ be so restricted that Φ will satisfy (20). It is found that the relation must hold:

$$\begin{aligned}F_z &- [F_{xp} + \mathfrak{p}F_{zp} + (\mathfrak{p}_x + \mathfrak{p}\mathfrak{p}_z)F_{pp} + (\mathfrak{q}_x + \mathfrak{p}\mathfrak{q}_z)F_{pq}] \\ &- [F_{yq} + \mathfrak{q}F_{zq} + (\mathfrak{p}_y + \mathfrak{q}\mathfrak{p}_z)F_{pq} + (\mathfrak{q}_y + \mathfrak{q}\mathfrak{q}_z)F_{qq}] = 0\end{aligned} \tag{22}$$

This condition is always satisfied when a field exists and for $\mathfrak{p}$, $\mathfrak{q}$ the functions:

$$\mathfrak{p} = \phi_x(x, y, \gamma), \qquad \mathfrak{q} = \phi_y(x, y, \gamma), \qquad z = \phi(x, y, \gamma)$$

are chosen. In fact, (22) then reduces to Lagrange's equation (18), written for $z = \phi$.

Replacing now in the expression for Φ, (21), $z, p, q, \mathfrak{p}, \mathfrak{q}$ by Z, P, Q, ϕ_x, ϕ_y respectively, and forming the integral $\bar{J} - J$, whose integrand is the difference:

$$F(x, y, Z, P, Q) - \Phi(x, y, Z, P, Q),$$

we have, in this difference, the Weierstrassian function $\mathcal{E}$, save as to a positive factor, $a = \cos \alpha$, where α denotes the acute angle that the normal to C at (x, y, Z) makes with the z-axis:

$$\begin{aligned}\mathcal{E} = a\{&F(x,y,Z,P,Q) - F(x,y,Z,\phi_x,\phi_y) - (P-\phi_x)F_p(x,y,Z,\phi_x,\phi_y)\\ &- (Q-\phi_y)F_q(x,y,Z,\phi_x,\phi_y)\}\\ = \tfrac{1}{2}a[&F_{pp}(\theta)(P-\phi_x)^2 + 2F_{pq}(\theta)(P-\phi_x)(Q-\phi_y) + F_{qq}(\theta)(Q-\phi_y)^2]\end{aligned} \tag{23}$$

Here, $\phi_x(x, y, \gamma)$, $\phi_y(x, y, \gamma)$ are regarded as functions of (x, y, Z), γ being determined by the relation

$$Z = \phi(x, y, \gamma),$$

and the expressions $F_{pp}(\theta)$, $F_{pq}(\theta)$, $F_{qq}(\theta)$ denote that the corresponding functions F_{pp}, etc., are formed for the arguments $(x, y, Z, \phi_x + \theta[P-\phi_x], \phi_y + \theta[Q-\phi_y])$, $0 < \theta < 1$.

Since the relation

$$F_{pq}^2 - F_{pp}F_{qq} < 0$$

holds at all points of $\mathfrak{R}$, the quadratic form in the last expression for $\mathcal{E}$ is a definite form, and hence, if $F_{pp} > 0$ (and consequently F_{qq} also > 0), the integral

$$\bar{J} - J = \iint\limits_{\mathfrak{T}} \mathcal{E} \sec a \, dxdy = \iint\limits_{\mathfrak{T}} \mathcal{E} dS$$

where dS denotes an element of the surface C, will always have a positive value except when

$$P - \phi_x = 0, \qquad Q - \phi_y = 0, \qquad \text{at all points of } \mathfrak{T}.$$

Since a field exists, it can be shown as in the earlier case that these equations are satisfied only when $\bar{C}$ and C coincide throughout.

Application. Minimum Surfaces. The problem is to span into a closed tortuous curve Γ a surface whose area shall be less than the area of any other such surface.

Suppose that the curve Γ has a continuously turning tangent, and that some of the surfaces spanned into Γ are such that, M being a properly chosen plane, an arbitrary normal to M meets the surface (inclusive of the boundary) in at most one point, and is not tangent to the surface at this point. Choose M as the (x, y) plane of a system of rectangular coordinates, (x, y, z). Then the area of the surface is given by the integral

$$\iint \sqrt{1 + p^2 + q^2} \, dxdy,$$

and this integral is to be made a minimum. Lagrange's equation is as follows:

$$(1+q^2)\frac{\partial^2 z}{\partial x^2} - 2pq\frac{\partial^2 z}{\partial x \partial y} + (1+p^2)\frac{\partial^2 z}{\partial y^2} = 0. \tag{24}$$

Thus if, of all the surfaces that can be spanned into Γ, which have at all interior points partial derivatives of the first and second order satisfying the conditions imposed on z at the beginning of this paragraph, there exists one which has a smaller area than any other, it must at all interior points satisfy (24). *Conversely, this condition is sufficient.* For, let the equation of this extremal, C, be

$$z = f(x, y).$$

Then a field exists about C extending indefinitely in both directions. As function ϕ we may choose the following.

$$\phi(x, y, \gamma) = f(x, y) + \gamma, \qquad \gamma_0 = 0, \qquad -\infty < \gamma < \infty.$$

The area of C is not merely less than that of other surfaces of the class considered lying in a restricted neighborhood of C, but it is less than the area of *any* such surface.

From this example we may generalize and say: *Lagrange's condition* (18), *combined with the conditions*

$$F_{pp} > 0, \qquad F_{pq}^2 - F_{pp}F_{qq} < 0,$$

is a sufficient condition for a minimum whenever the integrand F does not contain z explicitly.

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